José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

J.3312. *Proposed by Dumitru Bătineţu-Giurgiu and Claudia Nănuţi, Romania.* In any triangle ABC with area F the following inequality holds:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq 2\sqrt{3}\,F.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. By Radon's Inequality (a direct consequence of the Cauchy-Schwarz Inequality), for any positive real numbers, we have:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}=\frac{a^4}{a(b+c)}+\frac{b^4}{b(c+a)}+\frac{c^4}{c(a+b)}\geq\frac{(a^2+b^2+c^2)^2}{a(b+c)+b(c+a)+c(a+b)}$$

Expanding and grouping the denominator yields:

$$a(b+c)+b(c+a)+c(a+b)=2(ab+bc+ca)$$

Thus, we can write:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq\frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}$$

Using the well-known algebraic identity $a^2+b^2+c^2\geq ab+bc+ca$, we obtain:

$$\frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}\geq\frac{(a^2+b^2+c^2)(ab+bc+ca)}{2(ab+bc+ca)}=\frac{a^2+b^2+c^2}{2}$$

Therefore, we have established the lower bound:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq\frac{a^2+b^2+c^2}{2}$$

Weitzenböck's Inequality states that for any triangle with length sides a, b, c and area F:

$$a^2+b^2+c^2\geq 4\sqrt{3}\,F$$

Substituting Weitzenböck's Inequality back into our algebraic lower bound gives:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq\frac{a^2+b^2+c^2}{2}\geq\frac{4\sqrt{3}\,F}{2}=2\sqrt{3}\,F$$

Combining both parts completes the proof:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq 2\sqrt{3}\,F$$

Equality holds if and only if the triangle is equilateral ($a=b=c$).