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J.3311. *Proposed by Dumitru Bătineţu-Giurgiu and Claudia Nănuţi, Romania.* In any triangle ABC the following inequality holds:

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}\geq 2\sqrt{3}.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. Let Δ denote the area of the triangle ABC. The area can be expressed in terms of each side and its corresponding altitude:

$$\Delta=\frac{1}{2}ah_a=\frac{1}{2}bh_b=\frac{1}{2}ch_c$$

From these relations, we express the altitudes as:

$$h_a=\frac{2\Delta}{a},\quad h_b=\frac{2\Delta}{b},\quad h_c=\frac{2\Delta}{c}$$

Substituting these into the left-hand side of the target inequality gives:

$$\frac{a}{\frac{2\Delta}{a}}+\frac{b}{\frac{2\Delta}{b}}+\frac{c}{\frac{2\Delta}{c}}=\frac{a^2}{2\Delta}+\frac{b^2}{2\Delta}+\frac{c^2}{2\Delta}=\frac{a^2+b^2+c^2}{2\Delta}$$

Thus, the original inequality is logically equivalent to:

$$\frac{a^2+b^2+c^2}{2\Delta}\geq 2\sqrt{3}\Rightarrow a^2+b^2+c^2\geq 4\sqrt{3}\Delta$$

The inequality $a^2+b^2+c^2\geq 4\sqrt{3}\Delta$ is known as Weitzenböck's Inequality. For ease of reference, we will prove it by using the Law of Cosines and the area formula $\Delta=\frac{1}{2}ab\sin C$. Indeed, by the Law of Cosines, $c^2=a^2+b^2-2ab\cos C$. Substituting this yields:

$$\begin{aligned}a^2+b^2+c^2&=2a^2+2b^2-2ab\cos C\\&=2(a-b)^2+4ab-2ab\cos C\end{aligned}$$

We want to show that this expression is greater than or equal to $4\sqrt{3}\Delta=2\sqrt{3}\,ab\sin C$:

$$2(a-b)^2+4ab-2ab\cos C\geq 2\sqrt{3}\,ab\sin C$$

Since $2(a-b)^2\geq 0$, it suffices to establish that:

$$4ab-2ab\cos C\geq 2\sqrt{3}\,ab\sin C$$

Dividing both sides by $2ab$ simplifies the statement to:

$$2-\cos C\geq\sqrt{3}\sin C\Rightarrow 2\geq\cos C+\sqrt{3}\sin C$$

Using the linear combination trigonometric identity, the right-hand side can be rewritten as:

$$\cos C + \sqrt{3}\sin C = 2\left(\frac{1}{2}\cos C + \frac{\sqrt{3}}{2}\sin C\right) = 2\sin(C + 30^\circ)$$

Since the maximum possible value of the sine function is 1, then it is true that $2 \geq 2\sin(C + 30^\circ)$. Because Weitzenböck's inequality holds for all triangles, our algebraic transformation confirms that:

$$\frac{a}{h_a} + \frac{b}{h_b} + \frac{c}{h_c} \geq 2\sqrt{3}$$

Equality holds if and only if $a = b = c$, meaning the triangle ABC is equilateral.