

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{\csc^3 \frac{A}{2}}{\csc^2 \frac{B}{2} - \csc \frac{B}{2} \csc \frac{C}{2} + \csc^2 \frac{C}{2}} + \frac{\csc^3 \frac{B}{2}}{\csc^2 \frac{C}{2} - \csc \frac{C}{2} \csc \frac{A}{2} + \csc^2 \frac{A}{2}} + \frac{\csc^3 \frac{C}{2}}{\csc^2 \frac{A}{2} - \csc \frac{A}{2} \csc \frac{B}{2} + \csc^2 \frac{B}{2}} \geq 6$$

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We shall first prove :

$$s \left(r^2 ((4R + r)^2 - 2s^2) \right)^2 \stackrel{(*)}{\geq} 9s^2 r^4 (s(4Rr + r^2) - 6r^2 s)$$

$$\Leftrightarrow 4s^4 - (64R^2 + 68Rr - 41r^2)s^2 + (4R + r)^4 \stackrel{①}{\geq} 0$$

and $\because (s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove ①,

it suffices to prove : LHS of ① $\stackrel{?}{\geq} 4(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow 256R^4 + 256R^3r - 928R^2r^2 + 656Rr^3 - 99r^4 \stackrel{②}{\geq} (64R^2 - 60Rr - r^2)s^2$$

Finally, RHS of ② $\stackrel{\text{Gerretsen}}{\leq} (64R^2 - 60Rr - r^2)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq}$ LHS of ②

$$\Leftrightarrow 20t^3 - 73t^2 + 70t - 8 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \Leftrightarrow (t - 2)(20t^2 - 33t + 4) \stackrel{?}{\geq} 0$$

$\rightarrow$ true $\because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow$ ② $\Rightarrow$ ① $\Rightarrow$ (*) is true and now,

$$\begin{aligned} \sum_{\text{cyc}} \frac{\csc^3 \frac{A}{2}}{\csc^2 \frac{B}{2} - \csc \frac{B}{2} \csc \frac{C}{2} + \csc^2 \frac{C}{2}} &= \sum_{\text{cyc}} \frac{\frac{1}{\sin^3 \frac{A}{2}}}{\frac{1}{\sin^2 \frac{B}{2}} - \frac{1}{\sin \frac{B}{2} \sin \frac{C}{2}} + \frac{1}{\sin^2 \frac{C}{2}}} \\ &= \sum_{\text{cyc}} \frac{\left(\frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}} \right)^2}{\sin \frac{A}{2} \left(\sin^2 \frac{B}{2} - \sin \frac{B}{2} \sin \frac{C}{2} + \sin^2 \frac{C}{2} \right)} \stackrel{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\text{cyc}} \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}} \right)^2}{\sum_{\text{cyc}} \left(\sin \frac{A}{2} \left(\sin^2 \frac{B}{2} - \sin \frac{B}{2} \sin \frac{C}{2} + \sin^2 \frac{C}{2} \right) \right)} \\ &\geq \frac{\frac{2}{3} \left(\sum_{\text{cyc}} \sin \frac{A}{2} \right) \left(\sum_{\text{cyc}} \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}} \right)^2}{\left(\sum_{\text{cyc}} \sin \frac{A}{2} \right) \left(\sum_{\text{cyc}} \sin \frac{A}{2} \sin \frac{B}{2} \right) - 6 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \left(\because 1 \stackrel{\text{Jensen}}{\geq} \frac{2}{3} \left(\sum_{\text{cyc}} \sin \frac{A}{2} \right) \right) \stackrel{?}{\geq} 6 \\ &\Leftrightarrow \frac{(\sum_{\text{cyc}} x)(\sum_{\text{cyc}} x^2 y^2)^2}{x^2 y^2 z^2 \left((\sum_{\text{cyc}} x)(\sum_{\text{cyc}} xy) - 6xyz \right)} \stackrel{③}{\geq} 9 \left(x = \sin \frac{A}{2}, y = \sin \frac{B}{2}, z = \sin \frac{C}{2} \right) \end{aligned}$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s', R, r'$ (say)

and then : $\sum_{cyc} x = s', xyz = r'^2 s' \sum_{cyc} xy = 4R'r' + r'^2$ and

$$\sum_{cyc} x^2 y^2 = r'^2 ((4R' + r')^2 - 2s'^2) \text{ and so, } (*) \Leftrightarrow$$

$$\frac{s' \left(r'^2 ((4R' + r')^2 - 2s'^2) \right)^2}{s'^2 \cdot r'^4 (s' (4R'r' + r'^2) - 6r'^2 \cdot s')} \stackrel{?}{\geq} 9 \rightarrow \text{true via } (*) \Rightarrow \boxed{\boxed{(*) \text{ is true}}} \text{ and so,}$$

$$\frac{\csc^3 \frac{A}{2}}{\csc^2 \frac{B}{2} - \csc \frac{B}{2} \csc \frac{C}{2} + \csc^2 \frac{C}{2}} + \frac{\csc^3 \frac{B}{2}}{\csc^2 \frac{C}{2} - \csc \frac{C}{2} \csc \frac{A}{2} + \csc^2 \frac{A}{2}} + \frac{\csc^3 \frac{C}{2}}{\csc^2 \frac{A}{2} - \csc \frac{A}{2} \csc \frac{B}{2} + \csc^2 \frac{B}{2}} \geq 6 \quad \forall \triangle ABC, '' = '' \text{ iff } \triangle ABC \text{ is equilateral (QED)}$$