

ROMANIAN MATHEMATICAL MAGAZINE

In all $\triangle ABC$ such that $a \neq b \neq c$ the following relationship holds:

$$\frac{a^2 r_a}{(a-b)(a-c)} + \frac{b^2 r_b}{(b-a)(b-c)} + \frac{c^2 r_c}{(c-b)(c-a)} < \frac{27R^2}{4r}$$

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$$\begin{aligned} & \frac{a^2 F}{(S-a)(a-b)(a-c)} + \frac{b^2 F}{(S-b)(b-c)(b-a)} + \frac{c^2 F}{(S-c)(c-a)(c-b)} = \\ & = F \left(\frac{a^2}{(S-a)(a-b)(a-c)} + \frac{b^2}{(S-b)(b-c)(b-a)} + \frac{c^2}{(S-c)(c-a)(c-b)} \right) \end{aligned}$$

Let's prove that :

$$\begin{aligned} & \frac{a^2}{(s-a)(a-b)(a-c)} + \frac{b^2}{(s-b)(b-c)(b-a)} + \frac{c^2}{(s-c)(c-a)(c-b)} = \\ & = \frac{s^2}{(s-a)(s-b)(s-c)} \end{aligned}$$

So that for three distinct points a, b, c the Lagrange interpolation formula (expansion) for the function $f(x) = x^2$ is as follows:

$$x^2 = a^2 \cdot \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \cdot \frac{(x-a)(x-c)}{(b-a)(b-c)} + c^2 \cdot \frac{(x-a)(x-b)}{(c-b)(c-a)}$$

Showing that the expression is identically true. Let's find the derivative of both sides of the expression :

$$2x = a^2 \cdot \frac{(2x-(b+c))}{(a-b)(a-c)} + b^2 \cdot \frac{(2x-(a+c))}{(b-a)(b-c)} + c^2 \cdot \frac{(2x-(a+b))}{(c-b)(c-a)}$$

second derivative :

$$2 = \frac{2a^2}{(a-b)(a-c)} + \frac{2b^2}{(b-a)(b-c)} + \frac{2c^2}{(c-b)(c-a)}$$

Now let's prove that :

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$$1 = \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-b)(c-a)}$$

$$LHS = \frac{a^2(b-c) - b^2(a-c) - c^2(a-b)}{(a-b)(a-c)(b-c)}$$

$$\text{Numerator} = a^2(b-c) - b^2(a-c) - c^2(a-b) =$$

$$a^2(b-c) - b^2a + b^2c + c^2a - c^2b = a^2(b-c) - a(b^2 - c^2) + bc(b-c) =$$

$$(b-c)(a^2 - a(b+c) + bc) = (b-c)(a(a-b) - c(a-b)) = (b-c)(a-b)(a-c)$$

$$\text{So, } LHS = \frac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)} = 1$$

$$\frac{a^2 r_a}{(a-b)(a-c)} + \frac{b^2 r_b}{(b-a)(b-c)} + \frac{c^2 r_c}{(c-b)(c-a)} =$$

$$= F \cdot \frac{s^2}{(s-a)(s-b)(s-c)} = F \cdot \frac{s^2}{F^2} = \frac{s^3}{s \cdot r} = \frac{s^2}{r} \stackrel{\text{Mitrinović}}{\lesssim} \frac{27R^2}{4r}$$