

ROMANIAN MATHEMATICAL MAGAZINE

In triangles ABC such that $a \neq b \neq c$ the following relationship holds:

$$\frac{1}{27R^6} < \frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ba} < \frac{1}{1728r^6}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\begin{aligned} & \frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ba} = \\ &= \frac{1}{abc} \left(\frac{1}{(a-b)(a-c)a} + \frac{1}{(b-c)(b-a)b} + \frac{1}{(c-a)(c-b)c} \right) = \\ &= \frac{1}{abc} \left(\frac{-bc(b-c) - ac(c-a) - ab(a-b)}{(a-b)(b-c)(c-a)abc} \right) = \\ &= \frac{1}{(abc)^2} \left(\frac{-b^2c + c^2b + a^2c - c^2a - a^2b + b^2a}{(a-b)(b-c)(c-a)} \right) = \\ &= \frac{1}{(abc)^2} \left(\frac{bc(c-b) + a^2(c-b) + a(b^2 - c^2)}{(a-b)(b-c)(c-a)} \right) = \\ &\left\{ \begin{aligned} N &= bc(c-b) + a^2(c-b) + a(b^2 - c^2) = \\ &= (b-c)(c(a-b) - a(a-b)) = (b-c)(a-b)(c-a) \end{aligned} \right. \\ &= \frac{1}{(abc)^2} \left(\frac{(b-c)(a-b)(c-a)}{(a-b)(b-c)(c-a)} \right) = \frac{1}{(abc)^2} \\ &\frac{1}{(abc)^2} = \frac{1}{(4RF)^2} = \frac{1}{16R^2F^2} < \frac{1}{16 \cdot 4r^2 \cdot 27r^4} = \frac{1}{1728r^6} \text{ (RHS)} \\ &\frac{1}{(abc)^2} = \frac{1}{16R^2F^2} > \frac{1}{16R^2 \cdot \frac{27R^4}{16}} = \frac{1}{27R^6} \text{ (LHS)} \end{aligned}$$