ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{\tan\frac{A}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)}+\frac{\tan\frac{B}{2}}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{A}{2}\right)}+$$

$$+\frac{\tan\frac{C}{2}}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)}\geq\frac{54}{\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{r_a}=\sum_{cyc}\frac{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}{s\tan\frac{A}{2}}=\frac{4R}{s}.\frac{4R+r}{2R}$$

$$\therefore\left(\sum_{cyc}\frac{a}{r_a}\right)^3.\sum_{cyc}\frac{\tan\frac{A}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)}$$

$$=\frac{8(4R+r)^3}{s^3}.s\sum_{cyc}\frac{r_a(r_b+r_c)}{(r_a+r_b)(r_b+r_c)(r_c+r_a)}=\frac{8(4R+r)^3}{s^2}.\frac{2s^2}{\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)}$$

$$=\frac{16(4R+r)^3}{64R^3.\frac{s^2}{16R^2}}=\frac{4(4R+r)^3}{Rs^2}\overset{\text{Gerretsen}}{\geq}\frac{4(4R+r)^3}{R(4R^2+4Rr+3r^2)}\overset{?}{\geq}54$$

$$\Leftrightarrow 20t^3-12t^2-57t+2\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(20t^2+28t-1)\overset{?}{\geq}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\geq}2\therefore\sum_{cyc}\frac{\tan\frac{A}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)}\geq\frac{54}{\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)^3}\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$