ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship ho$l$ds :

$$\frac{r_a^3}{r_b^2-r_br_c+r_c^2}+\frac{r_b^3}{r_c^2-r_cr_a+r_a^2}+\frac{r_c^3}{r_a^2-r_ar_b+r_b^2}\geq 9r$$

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$$\sum_{cyc}\frac{r_a^3}{r_b^2-r_br_c+r_c^2}=\sum_{cyc}\frac{r_a^4}{r_ar_b^2-r_ar_br_c+r_c^2r_a}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc}r_a^2\right)^2}{\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-6r_ar_br_c}=\frac{((4R+r)^2-2s^2)^2}{(4R+r)s^2-6rs^2}\overset{?}{\geq}9r$$

$$\Leftrightarrow (4R+r)^4+4s^4-(64R^2+68Rr-41r^2)s^2\underset{(*)}{\underbrace{\overset{?}{\geq}}}0$$

$$\because 4(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore \text{ in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*)\overset{?}{\geq}4(s^2-16Rr+5r^2)^2\Leftrightarrow$$

$$256R^4+256R^3r-928R^2r^2+656Rr^3-99r^4\underset{(**)}{\underbrace{\overset{?}{\geq}}}(64R^2-60Rr-r^2)s^2$$

$$\text{Fin}a\text{lly}, (64R^2-60Rr-r^2)s^2\overset{\text{Gerretsen}}{\leq}(64R^2-60Rr-r^2)(4R^2+4Rr+3r^2)\overset{?}{\leq}$$

$$\text{LHS of } (**)\Leftrightarrow 20t^3-73t^2+70t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(20t^2-33t+4)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{r_a^3}{r_b^2-r_br_c+r_c^2}+\frac{r_b^3}{r_c^2-r_cr_a+r_a^2}+\frac{r_c^3}{r_a^2-r_ar_b+r_b^2}\geq 9r\ \forall\ \text{ABC},$$

$$''=''\text{ iff } \Delta\text{ ABC is equil}a\text{ter}a\text{l (QED)}$$