ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship ho$l$ds :

$$\frac{1}{\sin^2\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}+\frac{1}{\sin^2\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}+\sin^2\frac{A}{2}}\geq 4$$

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We shall first prove :

$$\frac{s^2\left((s^2-8Rr-2r^2)^2+(4Rr+r^2)^2+(s^2-8Rr-2r^2)(4Rr+r^2)\right)}{(s^2-8Rr-2r^2)r^2((4R+r)^2-2s^2)+r^2s(s^3-12Rrs)+(4Rr+r^2)^3-12Rr^3s^2+2r^2s^2(4Rr+r^2)-6r^4s^2}\overset{?}{\underset{(*)}{\geq}}9$$

$$\Leftrightarrow s^6-(12Rr-6r^2)s^4-r^2(4R+r)^2s^2+9r^3(4R+r)^3\overset{?}{\underset{①}{\geq}}0 \text{ and } \because$$

$$P=(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0 \therefore \text{ to prove ①, it suffices to prove : } LHS_{①}\overset{?}{\geq}P$$

$$\Leftrightarrow(36R-9r)s^4-r(864R^2-432Rr+81r^2)s^2+$$
$$4r^2(1168R^3-852R^2r+327Rr^2-29r^3)\overset{?}{\underset{②}{\geq}}0$$

and $\because (36R-9r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore$ to prove ②, suffices to prove :
LHS of ② $\overset{?}{\geq}(36R-9r)(s^2-16Rr+5r^2)^2\Leftrightarrow(288R^2-216Rr+9r^2)s^2$
$\overset{?}{\underset{③}{\geq}}r(4544R^3-4656R^2r+1032Rr^2-109r^3)$

Finally, LHS of ③ $\overset{\text{Gerretsen}}{\geq}(288R^2-216Rr+9r^2)(16Rr-5r^2)\overset{?}{\geq}$ RHS of ③
$\Leftrightarrow 4t^3-15t^2+12t+4\overset{?}{\geq}0\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)^2(4t+1)\overset{?}{\geq}0\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2$

$\Rightarrow ③\Rightarrow ②\Rightarrow ①\Rightarrow \boxed{\boxed{(*)\text{ is true}}}$ and now, since $1\overset{\text{Jensen}}{\geq}\frac{4}{9}\left(\sum_{cyc}\sin\frac{A}{2}\right)^2$

$\therefore$ in order to prove : $\sum_{cyc}\frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}\overset{?}{\geq}4$, it suffices to prove :

$$\left(\sum_{cyc}\sin\frac{A}{2}\right)^2\cdot\sum_{cyc}\frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}\overset{?}{\geq}9$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^2\cdot\sum_{cyc}\frac{(z^2+zx+x^2)(x^2+xy+y^2)}{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}\overset{?}{\geq}9\left(\begin{array}{c}x=\sin\frac{A}{2},y=\sin\frac{B}{2},\\ z=\sin\frac{C}{2}\end{array}\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^2\cdot\frac{\left(\sum_{cyc}x^2\right)^2+\left(\sum_{cyc}xy\right)^2+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)}{\left(\begin{array}{c}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2y^2\right)+xyz\left(\sum_{cyc}x^3\right)+\sum_{cyc}x^3y^3+\\ 2xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6x^2y^2z^2\end{array}\right)}\boxed{\overset{?}{\underset{(\bullet)}{\geq}}}9$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ *and* $Z+X-Y=2y>0\Rightarrow X,Y,Z$ *form sides of a* triangle $\Rightarrow X+Y>Z$, $Y+Z>X, Z+X>Y$ *with semiperimeter, circumradius* & inradius $=s',R,r'$ (*say*)

and then : $\sum_{cyc}x=s', xyz=r'^2s'\sum_{cyc}xy=4R'r'+r'^2, \sum_{cyc}x^2=s'^2-8R'r'-2r'^2$, $\sum_{cyc}x^3=s'^3-12R'r's', \sum_{cyc}x^2y^2=r'^2((4R'+r')^2-2s'^2)$ *and* $\sum_{cyc}x^3y^3=(4R'r'+r'^2)^3-12R'r'^3s'^2$ *and so*, $(\bullet)\Leftrightarrow$

$$\frac{s'^2\left((s'^2-8R'r'-2r'^2)^2+(4R'r'+r'^2)^2+(s'^2-8R'r'-2r'^2)(4R'r'+r'^2)\right)}{\begin{array}{c}(s'^2-8R'r'-2r'^2)r'^2((4R'+r')^2-2s'^2)+r'^2s'^{(s'^3-12R'r's')}+\\ (4R'r'+r'^2)^3-12R'r'^3s'^2+2r'^2s'^2(4R'r'+r'^2)-6r'^4s'^2\end{array}}\boxed{\overset{?}{\geq}}9$$

$\rightarrow$ true via $(*)\Rightarrow\boxed{\boxed{(\bullet)\text{ is true}}}$ *and so*,

$$\frac{1}{\sin^2\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}+\frac{1}{\sin^2\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}+\sin^2\frac{A}{2}}\geq 4\ \forall\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$