ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC such that $a \neq b \neq c$, the following relationship holds :

$$\frac{1}{\sin\frac{A}{2}\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}-\sin\frac{C}{2}\right)}+\frac{1}{\sin\frac{B}{2}\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)\left(\sin\frac{B}{2}-\sin\frac{A}{2}\right)}+\frac{1}{\sin\frac{C}{2}\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{C}{2}-\sin\frac{B}{2}\right)}>8$$

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$$\text{For } x \neq y \neq z, \sum_{cyc}\frac{1}{x(x-y)(x-z)}$$

$$=\frac{1}{xyz(x-y)(y-z)(z-x)}.\left(-yz(y-z)-zx(z-x)-xy(x-y)\right)$$

$$=-\frac{1}{xyz(x-y)(y-z)(z-x)}.\left(yz(y-z)-x(y-z)(y+z)+x^2(y-z)\right)$$

$$=-\frac{1}{xyz(x-y)(y-z)(z-x)}.(y-z)\left(y(z-x)-x(z-x)\right)$$

$$=\frac{1}{xyz(x-y)(y-z)(z-x)}.(y-z)(z-x)(x-y)$$

$$\therefore \sum_{cyc}\frac{1}{x(x-y)(x-z)}=\frac{1}{xyz} \text{ and via } x \equiv \sin\frac{A}{2}, y \equiv \sin\frac{B}{2}, z \equiv \sin\frac{C}{2},$$

$$\frac{1}{\sin\frac{A}{2}\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}-\sin\frac{C}{2}\right)}+\frac{1}{\sin\frac{B}{2}\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)\left(\sin\frac{B}{2}-\sin\frac{A}{2}\right)}+\frac{1}{\sin\frac{C}{2}\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{C}{2}-\sin\frac{B}{2}\right)}=\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{4R}{r}\overset{\text{Euler}}{>}8$$

(equality doesn't occur because $a \neq b \neq c$)

$$\therefore \frac{1}{\sin\frac{A}{2}\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}-\sin\frac{C}{2}\right)}+\frac{1}{\sin\frac{B}{2}\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)\left(\sin\frac{B}{2}-\sin\frac{A}{2}\right)}+\frac{1}{\sin\frac{C}{2}\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{C}{2}-\sin\frac{B}{2}\right)}>8 \ \forall \ \Delta \text{ ABC (QED)}$$