

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC such that $a \neq b \neq c$, the following relationship holds :

$$\frac{1}{\cot \frac{A}{2} \left(\cot \frac{A}{2} - \cot \frac{B}{2} \right) \left(\cot \frac{A}{2} - \cot \frac{C}{2} \right)} + \frac{1}{\cot \frac{B}{2} \left(\cot \frac{B}{2} - \cot \frac{C}{2} \right) \left(\cot \frac{B}{2} - \cot \frac{A}{2} \right)} +$$

$$+ \frac{1}{\cot \frac{C}{2} \left(\cot \frac{C}{2} - \cot \frac{A}{2} \right) \left(\cot \frac{C}{2} - \cot \frac{B}{2} \right)} < \frac{1}{3\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \text{For } x \neq y \neq z, \sum_{\text{cyc}} \frac{1}{x(x-y)(x-z)} \\ &= \frac{1}{xyz(x-y)(y-z)(z-x)} \cdot (-yz(y-z) - zx(z-x) - xy(x-y)) \\ &= -\frac{1}{xyz(x-y)(y-z)(z-x)} \cdot (yz(y-z) - x(y-z)(y+z) + x^2(y-z)) \\ &= -\frac{1}{xyz(x-y)(y-z)(z-x)} \cdot (y-z)(y(z-x) - x(z-x)) \\ &= \frac{1}{xyz(x-y)(y-z)(z-x)} \cdot (y-z)(z-x)(x-y) \\ &\therefore \sum_{\text{cyc}} \frac{1}{x(x-y)(x-z)} = \frac{1}{xyz} \text{ and via } x \equiv \cot \frac{A}{2}, y \equiv \cot \frac{B}{2}, z \equiv \cot \frac{C}{2}, \\ &\frac{1}{\cot \frac{A}{2} \left(\cot \frac{A}{2} - \cot \frac{B}{2} \right) \left(\cot \frac{A}{2} - \cot \frac{C}{2} \right)} + \frac{1}{\cot \frac{B}{2} \left(\cot \frac{B}{2} - \cot \frac{C}{2} \right) \left(\cot \frac{B}{2} - \cot \frac{A}{2} \right)} + \\ &\frac{1}{\cot \frac{C}{2} \left(\cot \frac{C}{2} - \cot \frac{A}{2} \right) \left(\cot \frac{C}{2} - \cot \frac{B}{2} \right)} = \frac{1}{\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}} = \frac{r_a r_b r_c}{s^3} = \frac{s^2 r}{s^3} = \frac{r}{s} \\ &\stackrel{\text{Mitrinovic}}{<} \frac{1}{3\sqrt{3}} \text{ (equality doesn't occur because } a \neq b \neq c) \\ &\therefore \frac{1}{\cot \frac{A}{2} \left(\cot \frac{A}{2} - \cot \frac{B}{2} \right) \left(\cot \frac{A}{2} - \cot \frac{C}{2} \right)} + \frac{1}{\cot \frac{B}{2} \left(\cot \frac{B}{2} - \cot \frac{C}{2} \right) \left(\cot \frac{B}{2} - \cot \frac{A}{2} \right)} + \\ &\frac{1}{\cot \frac{C}{2} \left(\cot \frac{C}{2} - \cot \frac{A}{2} \right) \left(\cot \frac{C}{2} - \cot \frac{B}{2} \right)} < \frac{1}{3\sqrt{3}} \quad \forall \Delta ABC \text{ (QED)} \end{aligned}$$