ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC such that $r_a \neq r_b \neq r_c$, the following relationship holds :

$$2s^2 < \frac{r_a^4}{(r_a - r_b)(r_a - r_c)} + \frac{r_b^4}{(r_b - r_c)(r_b - r_a)} + \frac{r_c^4}{(r_c - r_a)(r_c - r_b)} < \frac{27}{2}(3R^2 - 8r^2)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } x \neq y \neq z, \sum_{cyc} \frac{x^4}{(x-y)(x-z)}$$

$$= \frac{1}{(x-y)(y-z)(z-x)} \cdot \left(-x^4(y-z) - y^4(z-x) - z^4(x-y)\right)$$

$$\therefore \sum_{cyc} \frac{x^4}{(x-y)(x-z)} = -\frac{x^4(y-z) + y^4(z-x) + z^4(x-y)}{(x-y)(y-z)(z-x)} \to (1)$$

$$\text{Now, } x^4(y-z) + y^4(z-x) + z^4(x-y) =$$

$$x^4(y-z) - x(y-z)(y^3 + y^2z + yz^2 + z^3) + yz(y-z)(y^2 + yz + z^2)$$

$$= (y-z)\left(x(x^3 - y^3) - y^2z(x-y) - yz^2(x-y) - z^3(x-y)\right)$$

$$= (y-z)(x-y)(x^3 + x^2y + xy^2 - y^2z - yz^2 - z^3)$$

$$= (y-z)(x-y)\left((x-z)(x^2 + xz + z^2) + y(x-z)(x+z) + y^2(x-z)\right)$$

$$\therefore \sum_{cyc} x^4(y-z) = -(x-y)(y-z)(z-x)\left(\sum_{cyc} x^2 + \sum_{cyc} xy\right) \to (2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \sum_{cyc} \frac{x^4}{(x-y)(x-z)} = \sum_{cyc} x^2 + \sum_{cyc} xy \to (3) \text{ and via}$$

$$x \equiv r_a, y \equiv r_b, z \equiv r_c, \text{ we get :}$$

$$\frac{r_a^4}{(r_a - r_b)(r_a - r_c)} + \frac{r_b^4}{(r_b - r_c)(r_b - r_a)} + \frac{r_c^4}{(r_c - r_a)(r_c - r_b)} = \sum_{cyc} r_a^2 + \sum_{cyc} r_a r_b$$

$$= (4R + r)^2 - s^2 \overset{\substack{\text{Euler} \\ \text{and} \\ \text{Mitrinovic}}}{<} 16R^2 + 4R^2 + r^2 - 27r^2 = \frac{81R^2}{2} - \frac{41R^2}{2} - 26r^2$$

$$\overset{\text{Euler}}{<} \frac{81R^2}{2} - \frac{41.4r^2}{2} - 26r^2 = \frac{81R^2}{2} - 108r^2 = \frac{27}{2}(3R^2 - 8r^2)$$

$$(\text{equality doesn't occur because } r_a \neq r_b \neq r_c \Rightarrow a \neq b \neq c) \text{ and also,}$$

$$\sum_{cyc} \frac{r_a^4}{(r_a - r_b)(r_a - r_c)} = (4R + r)^2 - s^2 \overset{\text{Doucet or Trucht}}{>} 3s^2 - s^2 = 2s^2 \text{ and so,}$$

$$2s^2 < \frac{r_a^4}{(r_a - r_b)(r_a - r_c)} + \frac{r_b^4}{(r_b - r_c)(r_b - r_a)} + \frac{r_c^4}{(r_c - r_a)(r_c - r_b)} <$$

$$\frac{27}{2}(3R^2 - 8r^2) \ \forall \ \Delta \text{ ABC (QED)}$$