ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC such that $a \neq b \neq c$, the following relationship holds :

$$72r^2 < \frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} < 18R^2$$

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$$\text{For } x \neq y \neq z, \sum_{cyc} \frac{x^4}{(x-y)(x-z)}$$

$$= \frac{1}{(x-y)(y-z)(z-x)} . \left(-x^4(y-z) - y^4(z-x) - z^4(x-y)\right)$$

$$\therefore \sum_{cyc} \frac{x^4}{(x-y)(x-z)} = -\frac{x^4(y-z) + y^4(z-x) + z^4(x-y)}{(x-y)(y-z)(z-x)} \to ①$$

$$\text{Now, } x^4(y-z) + y^4(z-x) + z^4(x-y) =$$

$$x^4(y-z) - x(y-z)(y^3 + y^2z + yz^2 + z^3) + yz(y-z)(y^2 + yz + z^2)$$

$$= (y-z)\left(x(x^3 - y^3) - y^2z(x-y) - yz^2(x-y) - z^3(x-y)\right)$$

$$= (y-z)(x-y)(x^3 + x^2y + xy^2 - y^2z - yz^2 - z^3)$$

$$= (y-z)(x-y)\left((x-z)(x^2 + xz + z^2) + y(x-z)(x+z) + y^2(x-z)\right)$$

$$\therefore \sum_{cyc} x^4(y-z) = -(x-y)(y-z)(z-x)\left(\sum_{cyc} x^2 + \sum_{cyc} xy\right) \to ②$$

$$\therefore ① \text{ and } ② \Rightarrow \sum_{cyc} \frac{x^4}{(x-y)(x-z)} = \sum_{cyc} x^2 + \sum_{cyc} xy \to ③ \text{ and via}$$

$$x \equiv a, y \equiv b, z \equiv c, \text{we get :}$$

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} = \sum_{cyc} a^2 + \sum_{cyc} ab <$$

$$2\sum_{cyc} a^2 \overset{\text{Leibnitz}}{<} 18R^2 \text{ and also, } \sum_{cyc} \frac{a^4}{(a-b)(a-c)} = \sum_{cyc} a^2 + \sum_{cyc} ab > 2\sum_{cyc} ab$$

$$\overset{\text{Gordon}}{>} 8\sqrt{3}.rs \overset{\text{Mitrinovic}}{>} 8\sqrt{3}.r.3\sqrt{3}.r = 72r^2 \left(\begin{matrix}\text{equality doesn't occur because}\\ a \neq b \neq c\end{matrix}\right)$$

$$\text{and so, } 72r^2 < \frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} < 18R^2$$

$$\forall \Delta \text{ ABC (QED)}$$