ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\sqrt{3}r^2}{R^3} \le \frac{a}{(a+b)(a+c)} + \frac{b}{(b+c)(b+a)} + \frac{c}{(a+c)(b+c)} \le \frac{\sqrt{3}R^2}{32r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{(a+b)(a+c)} + \frac{b}{(b+c)(b+a)} + \frac{c}{(a+c)(b+c)} =$$

$$\frac{a}{(2p-c)(2p-b)} + \frac{b}{(2p-a)(2p-c)} + \frac{c}{(2p-a)(2p-b)} =$$

$$\frac{a(2p-a)+b(2p-b)+c(2p-c)}{(2p-a)(2p-c)(2p-b)} = \frac{4p^2-(a^2+b^2+c^2)}{(a+b)(b+c)(a+c)} \overset{AM-GM}{\le}$$

$$\frac{4p^2-2(p^2-r^2-4Rr)}{8abc} = \frac{2p^2+2r^2+8Rr}{32prR} \overset{Gerretsen,Mitrinovici,Euler}{\le}$$

$$\frac{8R^2+8Rr+6r^2+2r^2+8Rr}{32.3\sqrt{3}r.2r.r} = \frac{8(R^2+2rR+r^2)}{192\sqrt{3}r^3} \le$$

$$\frac{(R+r)^2}{24\sqrt{3}r^3} \le \frac{3R^2}{32\sqrt{3}r^3} = \frac{3\sqrt{3}R^2}{3.32r^3} = \frac{\sqrt{3}R^2}{32r^3} \quad (RHS)$$

$$\sum_{cyc}\frac{a}{(a+b)(a+c)} = \frac{4p^2-(a^2+b^2+c^2)}{(a+b)(b+c)(a+c)} =$$

$$\frac{2p^2+2r^2+8Rr}{(a+b)(b+c)(a+c)} \overset{AM-GM}{\ge} \frac{2p^2+2r^2+8Rr}{\left(\frac{(a+b)+(a+c)+(b+c)}{3}\right)^3} =$$

$$\frac{54(p^2+r^2+4Rr)}{64p^3} \overset{Gerretsen,Mitrinovici}{\ge} \frac{27(12Rr+3r^2+r^2+4Rr)}{32.\frac{81\sqrt{3}R^3}{8}} =$$

$$\frac{16Rr+4r^2}{12\sqrt{3}R^3} \overset{Euler}{\ge} \frac{36r^2}{12\sqrt{3}R^3} = \frac{\sqrt{3}r^2}{R^3} \quad (LHS)$$

$$Equality\ \ holds\ \ for: \quad a=b=c$$