ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC such that $r_a \neq r_b \neq r_c$, the following relationship holds :

$$\frac{64}{729R^6} < \frac{1}{(r_a-r_b)(r_a-r_c)r_a^2r_br_c} + \frac{1}{(r_b-r_c)(r_b-r_a)r_b^2r_cr_a} + \frac{1}{(r_c-r_a)(r_c-r_b)r_c^2r_ar_b} < \frac{1}{729r^6}$$

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$$\text{For } x \neq y \neq z, \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz}$$

$$= \frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.\left(-yz(y-z) - zx(z-x) - xy(x-y)\right)$$

$$= -\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.\left(yz(y-z) - x(y-z)(y+z) + x^2(y-z)\right)$$

$$= -\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.(y-z)\left(y(z-x) - x(z-x)\right)$$

$$= \frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.(y-z)(z-x)(x-y)$$

$$\therefore \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz} = \frac{1}{x^2y^2z^2} \text{ and via } x \equiv r_a, y \equiv r_b, z \equiv r_c,$$

$$\frac{1}{(r_a-r_b)(r_a-r_c)r_a^2r_br_c} + \frac{1}{(r_b-r_c)(r_b-r_a)r_b^2r_cr_a} + \frac{1}{(r_c-r_a)(r_c-r_b)r_c^2r_ar_b}$$

$$= \frac{1}{r_a^2r_b^2r_c^2} = \frac{1}{r^2s^4} \text{ and } \because \text{ via Mitrinovic and Euler, } \frac{1}{\frac{R^2}{4}.\frac{729R^4}{16}} < \frac{1}{r^2s^4} < \frac{1}{r^2.729r^4}$$

(equality doesn't occur because $r_a \neq r_b \neq r_c \Rightarrow a \neq b \neq c$)

$$\therefore \frac{64}{729R^6} < \frac{1}{(r_a-r_b)(r_a-r_c)r_a^2r_br_c} + \frac{1}{(r_b-r_c)(r_b-r_a)r_b^2r_cr_a} + \frac{1}{(r_c-r_a)(r_c-r_b)r_c^2r_ar_b} < \frac{1}{729r^6} \text{ (QED)}$$