ROMANIAN MATHEMATICAL MAGAZINE

In $a$ny Δ ABC such th$a$t $a \neq b \neq c$, the following relationship ho$l$ds :

$$\frac{1}{27R^6} < \frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ab} < \frac{1}{1728r^6}$$

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$$\text{For } x \neq y \neq z, \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz}$$

$$= \frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.\left(-yz(y-z) - zx(z-x) - xy(x-y)\right)$$

$$= -\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.\left(yz(y-z) - x(y-z)(y+z) + x^2(y-z)\right)$$

$$= -\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.(y-z)\left(y(z-x) - x(z-x)\right)$$

$$= \frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.(y-z)(z-x)(x-y)$$

$$\therefore \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz} = \frac{1}{x^2y^2z^2} \text{ and via } x \equiv a, y \equiv b, z \equiv c,$$

$$\frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ab} = \frac{1}{a^2b^2c^2}$$

$$= \frac{1}{16R^2r^2s^2} \text{ and } \because \text{ via Mitrinovic and Euler,}$$

$$\frac{1}{16R^2.\frac{R^2}{4}.\frac{27R^2}{4}} < \frac{1}{16R^2r^2s^2} < \frac{1}{64r^2.r^2.27r^2}$$

$$(\text{equality doesn't occur because } a \neq b \neq c) \therefore \frac{1}{27R^6} <$$

$$\frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ab} < \frac{1}{1728r^6} \text{ (QED)}$$