ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC such that $a \neq b \neq c$, the following relationship holds :

$$\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}<\frac{3\sqrt{3}R}{2r}$$

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$$\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}$$

$$=\frac{-rs}{(a-b)(b-c)(c-a)r^2s}\cdot\sum_{cyc}\big((b+c)(b-c)(s-b)(s-c)\big)$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\sum_{cyc}\Big((b^2-c^2)(-s^2+sa+bc)\Big)$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\left(\begin{array}{c}-s^2\sum_{cyc}(b^2-c^2)+s\sum_{cyc}\big(a(b^2-c^2)\big)+\\ \sum_{cyc}\big(bc(b^2-c^2)\big)\end{array}\right)$$

$$\therefore\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\left(s\sum_{cyc}\big(a(b^2-c^2)\big)+\sum_{cyc}\big(bc(b^2-c^2)\big)\right)\to (1)$$

$$\text{Now, }\sum_{cyc}\big(a(b^2-c^2)\big)=ab(b-a)+c^2(b-a)-c(b+a)(b-a)$$

$$=(b-a)\big(b(a-c)-c(a-c)\big)=(a-b)(b-c)(c-a)=\sum_{cyc}\big(a(b^2-c^2)\big)\to (2)$$

$$\sum_{cyc}\big(bc(b^2-c^2)\big)=bc(b-c)(b+c)-a(b-c)(b^2+bc+c^2)+a^3(b-c)$$

$$=(b-c)\Big(b^2(c-a)+bc(c-a)-a(c-a)(c+a)\Big)$$

$$=(b-c)(c-a)\big((b+a)(b-a)+c(b-a)\big)=-2s(a-b)(b-c)(c-a)$$

$$=\sum_{cyc}\big(bc(b^2-c^2)\big)\to (3)\text{ and so, }(1),(2)\text{ and }(3)\Rightarrow LHS=$$

$$\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\big(s(a-b)(b-c)(c-a)-2s(a-b)(b-c)(c-a)\big)$$

$$=\frac{s}{r}\overset{\text{Mitrinovic}}{<}\frac{3\sqrt{3}R}{2r};\text{ equality doesn't occur because }a\neq b\neq c\text{ and so,}$$

$$\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}<\frac{3\sqrt{3}R}{2r}\ \forall\,\Delta\text{ ABC (QED)}$$