ROMANIAN MATHEMATICAL MAGAZINE

$$\begin{cases}\text{In any } \Delta\, ABC \text{ such that } a \neq b \neq c, \text{the following relationship holds :}\\ \dfrac{a^2r_a}{(a-b)(a-c)}+\dfrac{b^2r_b}{(b-c)(b-a)}+\dfrac{c^2r_c}{(c-a)(c-b)}<\dfrac{27R^2}{4r}\\ \text{In any } \Delta\, ABC, \text{the following relationship holds : } abc \geq 24\sqrt{3}.r^3\end{cases}$$

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$$\frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)}=$$

$$=\frac{-rs}{(a-b)(b-c)(c-a)r^2s}.\sum_{cyc}\Big(a^2(b-c)(s-b)(s-c)\Big)=$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}.\sum_{cyc}\Big(a^2(b-c)(-s^2+sa+bc)\Big)=$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}.\left(\begin{matrix}-s^2\sum_{cyc}\Big(a^2(b-c)\Big)+s\sum_{cyc}\Big(a^3(b-c)\Big)+\\ abc\sum_{cyc}\big(a(b-c)\big)\end{matrix}\right)$$

$$\therefore \frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)}=$$

$$\frac{-1}{(a-b)(b-c)(c-a)r}.\left(-s^2\sum_{cyc}\Big(a^2(b-c)\Big)+s\sum_{cyc}\Big(a^3(b-c)\Big)\right)\rightarrow (1)$$

$$\text{Now, }\sum_{cyc}\Big(a^2(b-c)\Big)=a^2(b-c)+b^2(c-a)+c^2(a-b)=$$

$$a^2(b-c)+bc(b-c)-a(b-c)(b+c)=(b-c)\Big((a^2-ab)-(ac-bc)\Big)$$

$$=-(b-c)(a-b)(c-a)\Rightarrow -s^2\sum_{cyc}\Big(a^2(b-c)\Big)=s^2(a-b)(b-c)(c-a)\rightarrow (2)$$

$$\text{Again, }\sum_{cyc}\Big(a^3(b-c)\Big)=a^3(b-c)+b^3(c-a)+c^3(a-b)$$

$$=a^3(b-c)+bc(b-c)(b+c)-a(b-c)(b^2+bc+c^2)$$

$$=(b-c)(a^3+b^2c+bc^2-ab^2-abc-ac^2)$$

$$=(b-c)\Big(a(a-b)(a+b)-c^2(a-b)-bc(a-b)\Big)$$

$$=(b-c)(a-b)\Big((a^2-c^2)+(ab-bc)\Big)=-(b-c)(a-b)(c-a)(a+b+c)$$

$$\Rightarrow s\sum_{cyc}\Big(a^3(b-c)\Big)=-2s^2(a-b)(b-c)(c-a)\rightarrow (3)\ \therefore (1),(2)\text{ and }(3)\Rightarrow$$

$$\frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)}=$$

$$\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\left(-s^2(a-b)(b-c)(c-a)\right)=\frac{s^2}{r}\overset{\text{Mitrinovic}}{<}\frac{27R^2}{4r};$$

equality doesn't occur because $a \neq b \neq c$ and so,

$$\frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)}<\frac{27R^2}{4r}\ \forall\ \Delta\ ABC\ (QED)$$

$$Also,\ abc=4Rrs\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Euler}}}{\geq}8r^2.3\sqrt{3}.r=24\sqrt{3}.r^3\ \forall\ \Delta\ ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)