ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{a+h_b}{r_c}+\frac{b+h_c}{r_a}+\frac{c+h_a}{r_b}\ge\left(6+4\sqrt{3}\right).\frac{r}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{\text{AM-GM}}{\ge}3.\sqrt[3]{\frac{abc}{r_ar_br_c}}+3.\sqrt[3]{\frac{h_ah_bh_c}{r_ar_br_c}}=3.\sqrt[3]{\frac{4Rrs}{rs^2}}+3.\sqrt[3]{\frac{2r^2s^2}{Rrs^2}}\overset{\text{Mitrinovic}}{\ge}$$

$$3.\sqrt[3]{\frac{4}{s}.\frac{2s}{3\sqrt{3}}}+3.\sqrt[3]{\frac{8r^3}{R.4r^2}}\overset{\text{Euler}}{\ge}\frac{6r}{\sqrt{3}.r}+\frac{6r}{R}\overset{\text{Euler}}{\ge}\frac{12r}{\sqrt{3}.R}+\frac{6r}{R}=\left(6+4\sqrt{3}\right).\frac{r}{R}$$

$$\text{and so,}\frac{a+h_b}{r_c}+\frac{b+h_c}{r_a}+\frac{c+h_a}{r_b}\ge\left(6+4\sqrt{3}\right).\frac{r}{R}\ \forall\ \Delta\ \text{ABC},$$

$''=''$ iff Δ ABC is equilateral (QED)