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In any ΔABC the following relationship holds :

$$\frac{ah_b}{r_c} + \frac{bh_c}{r_a} + \frac{ch_a}{r_b} \geq \frac{12\sqrt{3}r^2}{R}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{LHS} &\stackrel{\text{AM-GM}}{\geq} 3 \cdot \sqrt[3]{\frac{abc \cdot h_a h_b h_c}{r_a r_b r_c}} = 3 \cdot \sqrt[3]{\frac{4Rrs \cdot 2r^2 s^2}{Rrs^2}} = 6 \cdot \sqrt[3]{r^2 s} \stackrel{\text{Mitrinovic}}{\geq} \frac{6\sqrt{3} \cdot r^2}{r} \\ &\stackrel{\text{Euler}}{\geq} 12\sqrt{3} \cdot \frac{r^2}{R} \text{ and so, } \frac{ah_b}{r_c} + \frac{bh_c}{r_a} + \frac{ch_a}{r_b} \geq 12\sqrt{3} \cdot \frac{r^2}{R} \quad \forall \Delta ABC, \\ &\quad " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$