ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin B\sin C}{\sin A}\cdot\frac{b^2h_c+c^2h_b}{h_ah_b\sin B+h_ah_c\sin C}\ge\frac{24r^2}{R}$$

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Solution by Jenish Rijal-Nepal

$$\sum_{cyc}\frac{\sin B\sin C}{\sin A}\cdot\frac{b^2h_c+c^2h_b}{h_ah_b\sin B+h_ah_c\sin C}=$$

$$=\sum_{cyc}\frac{\frac{b}{2R}\cdot\frac{c}{2R}}{\frac{a}{2R}}\cdot\frac{b^2\cdot\frac{2\Delta}{c}+c^2\cdot\frac{2\Delta}{b}}{\frac{2\Delta}{a}\cdot\frac{2\Delta}{b}\cdot\frac{b}{2R}+\frac{2\Delta}{a}\cdot\frac{2\Delta}{c}\cdot\frac{c}{2R}}=$$

$$=\sum_{cyc}\frac{bc}{a}\cdot\frac{\frac{b^2}{c}+\frac{c^2}{b}}{\frac{2\Delta}{a}+\frac{2\Delta}{a}}=\sum_{cyc}\frac{1}{4\Delta}(b^3+c^3)=\frac{1}{2\Delta}(a^3+b^3+c^3)\overbrace{\ge}^{AM-GM}$$

$$\ge\frac{1}{2\Delta}\cdot 3abc=\frac{1}{2\cdot\frac{abc}{4R}}\cdot 3abc=6R=\frac{6R^2}{R}\overbrace{\ge}^{Mitrinovic}\frac{6(2r)^2}{R}=\frac{24r^2}{R}$$

Equality holds if and only if the triangle is equilateral.