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In $\triangle ABC$ the following relationship holds:

$$ar_b + br_c + cr_c \geq 6F$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\begin{aligned} ar_b + br_c + cr_c &\stackrel{AM-GM}{\geq} 3\sqrt[3]{abc r_a r_b r_c} = 3\sqrt[3]{abc \cdot \frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}} = \\ &= 3\sqrt[3]{abc \cdot \frac{F^3 s}{s(s-a)(s-b)(s-c)}} = 3\sqrt[3]{4RF \cdot \frac{F^3 s}{F^2}} = 3\sqrt[3]{4RsF^2} = 3\sqrt[3]{4Rs \cdot r^2 s^2} = \\ &= 3s\sqrt[3]{4Rr^2} \stackrel{EULER}{\geq} 3s \cdot \sqrt[3]{8r \cdot r^2} = 3s \cdot \sqrt[3]{8r^3} = 3s \cdot \sqrt[3]{(2r)^3} = 3s \cdot 2r = \\ &= 6rs = 6F \end{aligned}$$

Equality holds for $a = b = c$.