

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$ar_a + br_b + cr_c \geq 6F$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\begin{aligned} ar_a + br_b + cr_c &= \sum_{cyc} a \cdot \frac{F}{s-a} = F \sum_{cyc} \frac{a}{s-a} = F \cdot \frac{2(2R-r)}{r} \geq \\ &\stackrel{EULER}{\geq} F \cdot \frac{2(4r-r)}{r} = F \cdot \frac{2 \cdot 3r}{r} = 6F. \text{ Equality holds for } a = b = c. \end{aligned}$$

Solution 2 by Daniel Sitaru – Romania

$$\begin{aligned} ar_a + br_b + cr_c &\stackrel{AM-GM}{\geq} 3\sqrt[3]{abcr_ar_br_c} = \\ &= 3 \cdot \sqrt[3]{abc \cdot \frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}} = 3 \sqrt[3]{\frac{abcF^3s}{s(s-a)(s-b)(s-c)}} = \\ &= 3 \sqrt[3]{\frac{4RF \cdot F^3s}{F^2}} = 3\sqrt[3]{4RsF^2} = 3\sqrt[3]{4Rs \cdot r^2s^2} = 3 \cdot s\sqrt[3]{4Rr^2} \stackrel{EULER}{\geq} 3s \cdot \sqrt[3]{4 \cdot 2r \cdot r^2} = \\ &= 3s \cdot \sqrt[3]{8r^3} = 3s \cdot 2r = 6rs = 6F. \text{ Equality holds for } a = b = c. \end{aligned}$$