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In $\triangle ABC$ the following relationship holds:

$$\frac{\sqrt{a} + \sqrt{b}}{\sqrt{h_a}} + \frac{\sqrt{b} + \sqrt{c}}{\sqrt{h_b}} + \frac{\sqrt{c} + \sqrt{a}}{\sqrt{h_c}} \geq 2^4 \sqrt[4]{108}$$

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Solution by Daniel Sitaru – Romania

$$\begin{aligned} \sum_{cyc} \frac{\sqrt{a} + \sqrt{b}}{\sqrt{h_a}} &\stackrel{AM-GM}{\geq} \sum_{cyc} \frac{2^4 \sqrt[4]{ab}}{\sqrt{h_a}} = 2 \sum_{cyc} \sqrt[4]{\frac{ab}{h_a^2}} \stackrel{AM-GM}{\geq} \\ &\geq 2 \cdot 3 \sqrt[3]{\sqrt[4]{\left(\frac{abc}{h_a h_b h_c}\right)^2}} = 6 \sqrt[6]{\frac{abc}{h_a h_b h_c}} = 6 \cdot \sqrt[6]{\frac{4RF}{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}}} = \\ &= 6 \cdot \sqrt[6]{\frac{4RF \cdot abc}{8F^3}} = 6 \sqrt[6]{\frac{16R^2 F^2}{8F^3}} = 6 \sqrt[6]{\frac{2R^2}{F}} = 6 \cdot \sqrt[6]{\frac{2R \cdot R}{r \cdot s}} = 6 \cdot \sqrt[6]{\frac{2R \cdot 2r}{r \cdot s}} = 6 \cdot \sqrt[6]{\frac{4R}{s}} \geq \\ &\stackrel{MITRINOVIC}{\geq} 6 \cdot \sqrt[6]{\frac{4R}{\frac{3\sqrt{3}}{2} R}} = 6 \sqrt[6]{\frac{8}{3\sqrt{3}}} = 6 \sqrt[6]{\left(\frac{2}{\sqrt{3}}\right)^3} = 6 \sqrt[6]{\frac{2}{\sqrt{3}}} = 6 \sqrt[6]{\frac{2\sqrt{3}}{3}} = 2 \cdot \sqrt[6]{9 \cdot \frac{2\sqrt{3}}{3}} \\ &= 2 \sqrt[6]{6\sqrt{3}} = 2 \sqrt[6]{\sqrt{36 \cdot 3}} = 2^4 \sqrt[4]{108} \end{aligned}$$

Equality holds for $a = b = c$.