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In ΔABC the following relationship holds:

$$\frac{a\cot\frac{A}{2}}{h_a}+\frac{b\cot\frac{B}{2}}{h_b}+\frac{c\cot\frac{C}{2}}{h_c}\ge 6$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a\cot\frac{A}{2}}{h_a}+\frac{b\cot\frac{B}{2}}{h_b}+\frac{c\cot\frac{C}{2}}{h_c}=\sum_{cyc}\frac{a\cot\frac{A}{2}}{h_a}=\sum_{cyc}\frac{a}{\frac{2F}{a}\cdot\tan\frac{A}{2}}=\frac{1}{2F}\sum_{cyc}\frac{a^2}{\tan\frac{A}{2}}\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{2F}\cdot 3\sqrt[3]{\frac{(abc)^2}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}=\frac{3}{2F}\sqrt[3]{\frac{(4Rrs)^2}{\frac{r}{s}}}=\frac{3}{2F}\cdot\sqrt[3]{\frac{16R^2r^2s^2\cdot s}{r}}=\frac{3s}{2F}\cdot 2\sqrt[3]{2R^2r}\ge$$

$$\overset{EULER}{\ge}\frac{6s}{2rs}\sqrt[3]{2\cdot(2r)^2\cdot r}=\frac{3}{r}\cdot 2r=6$$

Equality holds for $a=b=c$.