ROMANIAN MATHEMATICAL MAGAZINE

$$\text{In any acute } \Delta\, ABC \text{ with } \tan A + \tan C = 2\tan B$$

$$\text{the following relationship holds: } \cos A + \cos C \le \frac{3\sqrt{2}}{4}$$

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$$\tan A + \tan C = 2\tan B \Rightarrow \frac{\sin A}{\cos A} + \frac{\sin C}{\cos C} = 2\tan B \Rightarrow \frac{\sin(A+C)}{\cos A\cos C} = 2\tan B$$

$$\Rightarrow \frac{\sin B}{\cos A\cos C} = \frac{2\sin B}{\cos B} \Rightarrow 2\cos A\cos C = \cos B \Rightarrow -\cos B + \cos(A-C) = \cos B$$

$$\Rightarrow 2\cos^2\frac{A-C}{2} = 1 + 2\cos B \text{ and so, } (\cos A + \cos C)^2 = \left(2\sin^2\frac{B}{2}\right)(1+2\cos B)$$

$$= (1-\cos B)(1+2\cos B) = \frac{9}{8} - \left(\frac{1}{8} - \cos B + 2\cos^2 B\right) = \frac{9}{8} - \frac{(4\cos B - 1)^2}{8} \le \frac{9}{8}$$

$$\Rightarrow \boxed{\cos A + \cos C \le \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}} \left(\because 0 < A, C < \frac{\pi}{2} \Rightarrow \cos A + \cos C > 0\right)$$

$$\text{and } '' = '' \text{ iff } \boxed{\cos B = \frac{1}{4}} \text{ and consequently, iff } \cos(A-C) = \frac{1}{2},$$

$$\text{that is iff } (A-C) = \frac{\pi}{3} \text{ or } (C-A) = \frac{\pi}{3}$$

$$\text{When } A - C = \frac{\pi}{3}, \text{ then: } \tan A + \tan C = 2\tan B \Rightarrow \tan C + \frac{\sqrt{3} + \tan C}{1 - \sqrt{3}\tan C} = 2\sqrt{15}$$

$$\text{and solving this quadratic equation in } ''\tan C'', \text{ we get: } \boxed{\tan C = \frac{3\sqrt{5} - 6}{\sqrt{3}}}$$

$$\left(\because C < \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}\right) \text{ and consequently, via } \tan A + \tan C = 2\sqrt{15}, \text{ we get:}$$

$$\boxed{\tan A = \frac{3\sqrt{5} + 6}{\sqrt{3}}} \text{ and similarly, if } C - A = \frac{\pi}{3}, \text{ we get:}$$

$$\tan C = \frac{3\sqrt{5} + 6}{\sqrt{3}} \text{ and } \tan A = \frac{3\sqrt{5} - 6}{\sqrt{3}} \text{ and so, } \cos A + \cos C \le \frac{3\sqrt{2}}{4}$$

$$\forall \text{ acute } \Delta\, ABC \text{ with } \tan A + \tan C = 2\tan B,$$

$$'' = '' \text{ iff } \left\{\begin{pmatrix} A = \tan^{-1}\frac{3\sqrt{5}+6}{\sqrt{3}} \\ B = \cos^{-1}\frac{1}{4} \\ C = \tan^{-1}\frac{3\sqrt{5}-6}{\sqrt{3}} \end{pmatrix} \text{ or } \begin{pmatrix} A = \tan^{-1}\frac{3\sqrt{5}-6}{\sqrt{3}} \\ B = \cos^{-1}\frac{1}{4} \\ C = \tan^{-1}\frac{3\sqrt{5}+6}{\sqrt{3}} \end{pmatrix}\right\} \text{ (QED)}$$