ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}\le 2\cos\frac{A-B}{2}$$

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$$\sin\frac{3A}{2}+\sin\frac{3B}{2}=2\sin\frac{3A+3B}{4}\cos\frac{3A-3B}{4}\le 2\cos\frac{3A-3B}{4}$$

$$\left(\because 0<\sin\frac{3A+3B}{4}\le 1\ as\ 0<\frac{3A+3B}{4}<\frac{3\pi}{4}\right)\overset{?}{\le} 2\cos\frac{A-B}{2}=2\cos^2\frac{A-B}{4}-1$$

$$\Leftrightarrow 4\cos^3\theta-3\cos\theta\overset{?}{\le} 2\cos^2\theta-1\ \left(\theta=\frac{A-B}{4}\right)$$

$$\Leftrightarrow (t-1)(2t^2-1+2t^2+2t)\overset{?}{\le} 0\ (t=\cos\theta)\to \text{true}\because -\frac{\pi}{4}<\theta=\frac{A-B}{4}<\frac{\pi}{4}$$

$$\Rightarrow \frac{1}{\sqrt{2}}<t\le 1\Rightarrow 2t^2-1>0\therefore \sin\frac{3A}{2}+\sin\frac{3B}{2}\le 2\cos\frac{A-B}{2}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \cos\frac{A-B}{4}=1\ \wedge\ \sin\frac{3A+3B}{4}=1\Rightarrow \text{iff } A=B \text{ and subsequently iff}$$

$$\sin\frac{3A}{2}=1 \text{ and }\because 0<\frac{3A}{2}<\frac{3\pi}{2}\therefore \sin\frac{3A}{2}=1 \text{ iff }\frac{3A}{2}=\frac{\pi}{2}\Rightarrow \text{iff } A=\frac{\pi}{3} \text{ and so,}$$

$$\text{in conclusion}, ''='' \text{ iff } A=B=\frac{\pi}{3}\ (QED)$$