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In ΔABC the following relationship holds:

$$\frac{a}{h_a\sin^2\frac{A}{2}}+\frac{b}{h_b\sin^2\frac{B}{2}}+\frac{c}{h_c\sin^2\frac{C}{2}}\ge 8\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\sin^2\frac{A}{2}}+\frac{b}{h_b\sin^2\frac{B}{2}}+\frac{c}{h_c\sin^2\frac{C}{2}}=\sum_{cyc}\frac{a}{h_a\cdot\sin^2\frac{A}{2}}=\sum_{cyc}\frac{a}{\frac{2F}{a}\cdot\frac{(s-b)(s-c)}{bc}}=$$

$$=\frac{abc}{2F}\sum_{cyc}\frac{a}{(s-b)(s-c)}=\frac{4RF}{2F}\cdot\frac{1}{(s-a)(s-b)(s-c)}\sum_{cyc}a(s-a)=$$

$$=\frac{2Rs}{s(s-a)(s-b)(s-c)}\cdot 2r(4R+r)\overset{EULER}{\ge}\frac{2Rs}{F^2}\cdot 2r(8r+r)=\frac{4Rrs\cdot 9r}{F\cdot rs}=$$

$$=\frac{36Rr}{F}=\frac{36Rr}{rs}=\frac{36R}{s}\overset{MITRINOVIC}{\ge}\frac{36R}{\frac{3\sqrt{3}}{2}R}=\frac{72}{3\sqrt{3}}=\frac{24}{\sqrt{3}}=\frac{24\sqrt{3}}{3}=8\sqrt{3}$$

Equality holds for $a=b=c$.