ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a^2}{r_ar_b}+\frac{b^2}{r_br_c}+\frac{c^2}{r_cr_a}\ge 4$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^2}{r_ar_b}+\frac{b^2}{r_br_c}+\frac{c^2}{r_cr_a}\ge 3\sqrt[3]{\left(\frac{abc}{r_ar_br_c}\right)^2}=3\sqrt[3]{\left(\frac{4RF}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{4RF}{\frac{F^3\cdot s}{s(s-a)(s-b)(s-c)}}\right)^2}=3\sqrt[3]{\left(\frac{4RF\cdot F^2}{F^3s}\right)^2}=3\sqrt[3]{\left(\frac{4R}{s}\right)^2}\overset{MITRINOVIC}{\ge}$$

$$\ge 3\sqrt[3]{\left(\frac{4\cdot\frac{2}{3\sqrt{3}}s}{s}\right)^2}=3\sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2}=3\sqrt[3]{\frac{64}{27}}=3\cdot\frac{4}{3}=4$$

Equality holds for $a=b=c$.