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Prove that in any ΔABC holds:

$$\sin^6\frac{A}{2}+\sin^6\frac{B}{2}+\sin^6\frac{C}{2}\geq\frac{3}{64}$$

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Let A,B,C be the internal angles of ΔABC. Using the fundamental trigonometric identity:

$$\sum\sin^2\frac{A}{2}=1-2\prod\sin\frac{A}{2}$$

By Euler's Inequality ($R\geq 2r$):

$$\prod\sin\frac{A}{2}=\frac{r}{4R}\overset{\text{Euler}}{\leq}\frac{1}{8}$$

Therefore:

$$\sum\sin^2\frac{A}{2}=1-2\prod\sin\frac{A}{2}\overset{\text{Euler}}{\geq}1-2\left(\frac{1}{8}\right)=\frac{3}{4}$$

Now, applying the Power Mean Inequality:

$$\sum_{cyc}\sin^6\frac{A}{2}\geq 3\cdot\left(\frac{\sum_{cyc}\sin^2\frac{A}{2}}{3}\right)^3\geq 3\cdot\frac{1}{27}\cdot\left(\frac{3}{4}\right)^3=\frac{3}{64}$$

Equality holds if and only if $A=B=C=60^\circ$ (ΔABC is equilateral).