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In ΔABC the following relationship holds:

$$(h_a)^{\cos(B)+\cos(C)}.(h_b)^{\cos(A)+\cos(C)}.(h_c)^{\cos(B)+\cos(A)} \le \left(\frac{3R}{2}\right)^3$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$In\ \Delta ABC\ wlog:\ a \le b \le c,\qquad h_a \ge h_b \ge h_c$$

$$\rightarrow \begin{cases} \cos(A)+\cos(B) \ge \cos(A)+\cos(C) \ge \cos(C)+\cos(B) \\ \cos(A) \ge \cos(B) \ge \cos(C) \end{cases}$$

$$(h_a)^{\cos(B)+\cos(C)}.(h_b)^{\cos(A)+\cos(C)}.(h_c)^{\cos(B)+\cos(A)} \overset{Weighted\ AM-GM}{\overbrace{\le}}$$

$$\left(\frac{h_a(\cos(B)+\cos(C))+h_b(\cos(A)+\cos(C))+h_c(\cos(B)+\cos(A))}{2(\cos(A)+\cos(B)+\cos(C))}\right)^{2\sum\cos(A)} \overset{Chebyshev}{\overbrace{\le}}$$

$$\le \left(\frac{\frac{1}{3}(h_a+h_b+h_c)\big(2(\cos(A)+\cos(B)+\cos(C))\big)}{2(\cos(A)+\cos(B)+\cos(C))}\right)^{2\sum\cos(A)} =$$

$$= \left(\frac{1}{3}(h_a+h_b+h_c)\right)^{2\sum\cos(A)} \le \left(\frac{1}{3}.\frac{9R}{2}\right)^{2\sum\cos(A)} \overset{Jensen}{\overbrace{\le}} \left(\frac{3R}{2}\right)^3$$

$$Equality\ holds\ if\ a=b=c$$