ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{r_a \sin^2 A}+\frac{b}{r_b \sin^2 B}+\frac{c}{r_c \sin^2 C}\ge \frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{r_a \sin^2 A}+\frac{b}{r_b \sin^2 B}+\frac{c}{r_c \sin^2 C}=\sum_{cyc}\frac{a}{r_a \sin^2 A}=\sum_{cyc}\frac{a}{\frac{F}{s-a}\cdot\frac{a^2}{4R^2}}=$$

$$=\frac{4R^2}{F}\sum_{cyc}\frac{s-a}{a}=\frac{4R^2}{F}\left(s\sum\frac{1}{a}-3\right)=$$

$$=\frac{4R^2}{rs}\left(s\cdot\frac{ab+bc+ca}{abc}-3\right)=\frac{4R^2}{rs}\left(s\frac{s^2+r^2+4Rr}{4Rrs}-3\right)=$$

$$=\frac{4R^2}{rs}\cdot\frac{s^2+r^2+4Rr-12Rr}{4Rr}=\frac{R}{r^2s}\cdot(s^2+r^2-8Rr)\ge$$

$$\overset{GERRETSEN}{\ge}\frac{R}{R^2S}(16Rr-5r^2+r^2-8Rr)=\frac{R}{r^2s}(8Rr-4r^2)=\frac{R(8R-4r)}{rs}\overset{EULER}{\ge}$$

$$\ge\frac{R(16r-4r)}{rs}\overset{MITRINOVIC}{\ge}\frac{\frac{2}{3\sqrt{3}}s\cdot 12r}{rs}=\frac{24}{3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.