ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$192\sqrt{3}r^3 \le (a+b)(b+c)(c+a) \le 24\sqrt{3}R^3$$

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We will use the known result:

$$(a+b)(b+c)(c+a) = 2s(s^2+r^2+2Rr)$$

We must prove that:

$$192\sqrt{3}r^3 \le 2s(s^2+r^2+2Rr) \le 24\sqrt{3}R^3$$

$$2s(s^2+r^2+2Rr) \overset{MITRINOVIC-GERRETSEN}{\le} 2\cdot\frac{3\sqrt{3}R}{2}(4R^2+4Rr+3r^2+r^2+2Rr) =$$

$$= 3\sqrt{3}R(4R^2+6Rr+4r^2) \le 3\sqrt{3}R\left(4R^2+6R\cdot\frac{R}{2}+4\cdot\frac{R^2}{4}\right) =$$

$$= 3\sqrt{3}R(4R^2+4R^2) = 24\sqrt{3}R^3$$

$$2s(s^2+r^2+2Rr) \overset{MITRINOVIC-EULER}{\ge} 2\cdot 3\sqrt{3}r(27r^2+r^2+2\cdot 2r\cdot r) =$$

$$= 6\sqrt{3}r(28r^2+4r^2) = 6\sqrt{3}r\cdot 32r^2 = 192\sqrt{3}r^3$$

Equality holds for $a=b=c$.