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In acute ΔABC the following relationship holds:

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\ge\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{abc}{h_ah_bh_c\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2}}=$$

$$=3\sqrt[3]{\frac{abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\cdot\left(\frac{s}{4R}\right)^2}}=3\sqrt[3]{\frac{(abc)^2\cdot 16R^2}{8F^3\cdot s^2}}=3\sqrt[3]{\frac{16R^2F^2\cdot 2R^2}{F^3s^2}}=$$

$$=3\sqrt[3]{\frac{32R^4}{Fs^2}}\overset{EULER}{\ge}3\sqrt[3]{\frac{32\cdot 2r\cdot R^3}{rs^3}}=3\sqrt[3]{64\left(\frac{R}{s}\right)^3}=\frac{12R}{s}\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{12\cdot\frac{2}{3\sqrt{3}}s}{s}=\frac{24}{3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.