ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{ab}{\sin C}+\frac{bc}{\sin A}+\frac{ca}{\sin B}\geq 24\sqrt{3}r^2$$

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Solution by Daniel Sitaru – Romania

$$\frac{ab}{\sin C}+\frac{bc}{\sin A}+\frac{ca}{\sin B}=\sum_{cyc}\frac{ab}{\sin C}=\sum_{cyc}\frac{ab}{\frac{c}{2R}}=2R\sum\frac{ab}{c}\overset{AM-GM}{\geq}2R\cdot 3\sqrt[3]{\frac{ab}{c}\cdot\frac{bc}{a}\cdot\frac{ca}{b}}=$$

$$=6R\sqrt[3]{abc}=6R\sqrt[3]{4Rrs}\overset{EULER}{\geq}6R\cdot\sqrt[3]{4\cdot 2r\cdot rs}=12R\sqrt[3]{r^2s}\overset{MITRINOVIC}{\geq}$$

$$\geq 12R\sqrt[3]{r^2\cdot 3\sqrt{3}r}\overset{EULER}{\geq}12\cdot 2r\cdot\sqrt[3]{\left(r\sqrt{3}\right)^3}=24r\cdot\sqrt{3}r=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.