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In ΔABC the following relationship holds:

$$\frac{ab}{\cos\frac{C}{2}}+\frac{bc}{\cos\frac{A}{2}}+\frac{ca}{\cos\frac{B}{2}}\ge 24\sqrt{3}r^2$$

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Solution by Daniel Sitaru – Romania

$$\frac{ab}{\cos\frac{C}{2}}+\frac{bc}{\cos\frac{A}{2}}+\frac{ca}{\cos\frac{B}{2}}=\sum_{cyc}\frac{ab}{\cos\frac{C}{2}}\overset{CEBYSHEV}{\ge}\frac{1}{3}\sum_{cyc}ab\cdot\sum_{cyc}\frac{1}{\cos\frac{C}{2}}\overset{JENSEN}{\ge}$$

$$\ge\frac{1}{3}\cdot\sum_{cyc}ab\cdot 3\cdot\frac{1}{\cos\left(\frac{A+B+C}{6}\right)}\overset{AM-GM}{\ge}\frac{1}{\cos\frac{\pi}{6}}\cdot\sum_{cyc}ab\ge\frac{1}{\frac{\sqrt{3}}{2}}\cdot 3\sqrt[3]{(abc)^2}=$$

$$=\frac{6}{\sqrt{3}}\cdot\sqrt[3]{(4RF)^2}=\frac{6\sqrt{3}}{3}\cdot\sqrt[3]{(4Rrs)^2}\overset{EULER}{\ge}2\sqrt{3}\cdot\sqrt[3]{(8r^2s)^2}\ge$$

$$\overset{MITRINOVIC}{\ge}2\sqrt{3}\cdot\sqrt[3]{\left(8r^2\cdot 3\sqrt{3}r\right)^2}=2\sqrt{3}\cdot\sqrt[3]{64r^6\cdot 27}=2\sqrt{3}\cdot 4\cdot 3\cdot r^2=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.