ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin A}{h_a}\ge\frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{\sin A}{h_a}=\sum_{cyc}\frac{\frac{a}{2R}}{\frac{2F}{a}}=\frac{1}{4RF}\sum_{cyc}a^2=\frac{1}{4Rrs}\cdot 2(s^2-r^2-4Rr)\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{1}{2Rr\cdot\frac{3\sqrt{3}R}{2}}(s^2-r^2-4Rr)=\frac{1}{3\sqrt{3}R^2r}(s^2-r^2-4Rr)\ge$$

$$\overset{GERRETSEN}{\ge}\frac{1}{3\sqrt{3}R^2r}(16Rr-5r^2-r^2-4Rr)=\frac{6r(2R-r)}{3\sqrt{3}R^2r}=$$

$$=\frac{2(2R-r)}{\sqrt{3}R^2}\ge\frac{\sqrt{3}}{R}\Leftrightarrow 2(2R-r)\ge 3r,\ 4R-2r\ge 3R\Leftrightarrow R\ge 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.