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In ΔABC the following relationship holds:

$$\frac{a}{h_a \sin\frac{A}{2}}+\frac{b}{h_b \sin\frac{B}{2}}+\frac{c}{h_c \sin\frac{C}{2}}\geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a \sin\frac{A}{2}}+\frac{b}{h_b \sin\frac{B}{2}}+\frac{c}{h_c \sin\frac{C}{2}}=\sum_{cyc}\frac{a}{h_a \sin\frac{A}{2}}=\sum_{cyc}\frac{a}{\frac{2F}{a}\sin\frac{A}{2}}=\frac{1}{2F}\sum_{cyc}\frac{a^2}{\sin\frac{A}{2}}\overset{CEBYSEV}{\geq}$$

$$\geq\frac{1}{2F}\cdot\frac{1}{3}\sum_{cyc}a^2\sum_{cyc}\frac{1}{\sin\frac{A}{2}}\overset{JENSEN}{\geq}\frac{1}{6F}\cdot\sum_{cyc}a^2\cdot\frac{3}{\sin\left(\frac{A+B+C}{6}\right)}=\frac{1}{2F}\sum_{cyc}a^2\cdot\frac{1}{\frac{1}{2}}=$$

$$=\frac{1}{rs}\cdot 2(s^2-r^2-4Rr)\overset{GERRETSEN}{\geq}\frac{2}{rs}(16Rr-5r^2-r^2-4Rr)=\frac{2(12Rr-6r^2)}{rs}$$

$$=\frac{2(12R-6r)}{s}=\frac{12(2R-r)}{s}\geq 4\sqrt{3}\Leftrightarrow\Leftrightarrow 3(2R-r)\geq s\sqrt{3}\ \text{(to prove)}$$

$$s\sqrt{3}\overset{DOUCET}{\leq}4R+r\leq 3(2R-r),\ 4R+r\leq 6R-3r,\ R\geq 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.