ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cos A\cos B\cos C \le \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

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We use:

$$\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}, \qquad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$$

We must prove:

$$\frac{s^2-(2R+r)^2}{4R^2} \le \frac{r}{4R}, \qquad s^2-(2R+r)^2 \le Rr$$

$$s^2 \le 4R^2+4Rr+r^2+Rr = 4R^2+5Rr+r^2$$

$$s^2 \overset{GERRETSEN}{\le} 4R^2+4Rr+3r^2 \le 4R^2+5Rr+r^2 \Leftrightarrow$$

$$\Leftrightarrow 3r^2 \le Rr+r^2 \Leftrightarrow Rr \ge 2r^2 \Leftrightarrow R \ge 2r \text{ (Euler)}$$

Equality holds for $a=b=c$.