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In $\triangle ABC$ the following relationship holds:

$$\sqrt{a \cos \frac{A}{2}} + \sqrt{b \cos \frac{B}{2}} + \sqrt{c \cos \frac{C}{2}} \leq \frac{3\sqrt{6R}}{2}$$

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Solution by Daniel Sitaru – Romania

$$\begin{aligned} \sqrt{a \cos \frac{A}{2}} + \sqrt{b \cos \frac{B}{2}} + \sqrt{c \cos \frac{C}{2}} &\stackrel{CBS}{\leq} \sqrt{\sum_{cyc} a \cdot \sum_{cyc} \cos \frac{A}{2}} \stackrel{JENSEN}{\leq} \\ &\leq \sqrt{2s \cdot 3 \cos \left(\frac{A+B+C}{6} \right)} = \sqrt{6s \cdot \cos \frac{\pi}{6}} = \sqrt{6s \cdot \frac{\sqrt{3}}{2}} = \sqrt{3\sqrt{3}s} \end{aligned}$$

Remains to prove:

$$\sqrt{3\sqrt{3}s} \leq \frac{3\sqrt{6R}}{2} \Leftrightarrow 3\sqrt{3}s \leq \frac{54R}{2} \Leftrightarrow 3\sqrt{3}s \leq 27R \Leftrightarrow s \leq \frac{9R}{\sqrt{3}} \Leftrightarrow s \leq \frac{3\sqrt{3}}{2}R \quad (\text{MITRINOVIC})$$

Equality holds for $a = b = c$.