ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c^3}+\frac{b^4+c^4}{h_a^3}+\frac{c^4+a^4}{h_b^3}\ge 32r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c^3}+\frac{b^4+c^4}{h_a^3}+\frac{c^4+a^4}{h_b^3}=\sum_{cyc}\frac{a^4+b^4}{h_c^3}\ge$$

$$\overset{AM-GM}{\ge} 2\sum_{cyc}\frac{a^2b^2}{h_c^3}\overset{AM-GM}{\ge} 2\cdot 3\sqrt[3]{\frac{(abc)^4}{(h_ah_bh_c)^3}}=\frac{6abc}{h_ah_bh_c}\sqrt[3]{abc}=$$

$$=\frac{6abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\sqrt[3]{4RF}=\frac{6(abc)^2}{8F^3}\sqrt[3]{4Rrs}\ge$$

$$\overset{EULER}{\ge}\frac{6\cdot 16R^2F^2}{8F^3}\sqrt[3]{8r^2s}\overset{MITRINOVIC}{\ge}\frac{12R^2}{F}\sqrt[3]{8r^2\cdot 3\sqrt{3}r}=$$

$$=\frac{12R^2}{rs}\cdot 2\sqrt{3}r=\frac{24\sqrt{3}R^2}{s}\overset{MITRINOVIC}{\ge}\frac{24\sqrt{3}\cdot R^2}{\frac{3\sqrt{3}}{2}R}=16R\overset{EULER}{\ge}16\cdot 2r=32r$$

Equality holds for $a=b=c$.