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In $\triangle ABC$ the following relationship holds:

$$\frac{h_a h_b}{r_c^2} + \frac{h_b h_c}{r_a^2} + \frac{h_c h_a}{r_b^2} \geq 12 \frac{r^2}{R^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\begin{aligned} \frac{h_a h_b}{r_c^2} + \frac{h_b h_c}{r_a^2} + \frac{h_c h_a}{r_b^2} &\geq 3 \sqrt[3]{\left(\frac{h_a h_b h_c}{r_a r_b r_c}\right)^2} = \\ &= 3 \sqrt[3]{\left(\frac{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}}{\frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}}\right)^2} = 3 \sqrt[3]{\left(\frac{8(s-a)(s-b)(s-c)}{abc}\right)^2} = \\ &= 3 \sqrt[3]{\left(\frac{8s(s-a)(s-b)(s-c)}{sabc}\right)^2} = 3 \sqrt[3]{\left(\frac{8F^2}{s \cdot 4RF}\right)^2} = \\ &= 3 \sqrt[3]{\left(\frac{2F}{Rs}\right)^2} = 3 \sqrt[3]{\left(\frac{2rs}{RS}\right)^2} = 3 \sqrt[3]{4 \left(\frac{r}{R}\right)^2} \geq 12 \left(\frac{r}{R}\right)^2 \Leftrightarrow \\ 27 \cdot 4 \left(\frac{r}{R}\right)^2 &\geq 12^3 \cdot \left(\frac{r}{R}\right)^6 \Leftrightarrow 3^3 \cdot 4 \geq 3^3 \cdot 4^3 \cdot \left(\frac{r}{R}\right)^4 \Leftrightarrow \frac{1}{16} \geq \left(\frac{r}{R}\right)^4 \Leftrightarrow \frac{1}{2} \geq \frac{r}{R} \Leftrightarrow R \geq 2r \\ \text{Equality holds for } a = b = c. \end{aligned}$$