ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a h_b}{r_a r_b}+\frac{h_b h_c}{r_b r_c}+\frac{h_c h_a}{r_c r_a}\ge\frac{12r^2}{R^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a h_b}{r_a r_b}+\frac{h_b h_c}{r_b r_c}+\frac{h_c h_a}{r_c r_a}\overset{AM-GM}{\ge}3\sqrt[3]{\left(\frac{h_a h_b h_c}{r_a r_b r_c}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}\right)^2}=3\sqrt[3]{\left(\frac{8(s-a)(s-b)(s-c)}{abc}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{8s(s-a)(s-b)(s-c)}{sabc}\right)^2}=3\sqrt[3]{\left(\frac{8F^2}{s\cdot 4RF}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{2F}{Rs}\right)^2}=3\sqrt[3]{\left(\frac{2rs}{Rs}\right)^2}=3\sqrt[3]{4\left(\frac{r}{R}\right)^2}\ge 12\left(\frac{r}{R}\right)^2\Leftrightarrow 27\cdot 4\left(\frac{r}{R}\right)^2\ge 12^3\cdot\left(\frac{r}{R}\right)^6$$

$$3^3\cdot 4\ge 3^3\cdot 4^3\cdot\left(\frac{r}{R}\right)^4,\ \frac{1}{16}\ge\left(\frac{r}{R}\right)^4\Leftrightarrow\frac{1}{2}\ge\frac{r}{R}\Leftrightarrow R\ge 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.