ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a}{r_b}+\frac{h_b}{r_c}+\frac{h_c}{r_a}\geq\frac{6r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{r_b}+\frac{h_b}{r_c}+\frac{h_c}{r_a}=\sum_{cyc}\frac{\frac{2F}{a}}{\frac{F}{s-b}}=2\sum_{cyc}\frac{s-b}{a}\geq$$

$$\overset{AM-GM}{\geq}2\cdot 3\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{abc}}=6\sqrt[3]{\frac{s(s-a)(s-b)(s-c)}{s\cdot 4RF}}=$$

$$=6\sqrt[3]{\frac{F^2}{4sRF}}=6\sqrt[3]{\frac{F}{4sR}}=6\sqrt[3]{\frac{rs}{4sR}}=6\sqrt[3]{\frac{r}{4R}}\geq\frac{6r}{R}\Leftrightarrow$$

$$\Leftrightarrow\frac{r}{4R}\geq\left(\frac{r}{R}\right)^3\Leftrightarrow\frac{1}{4}\geq\frac{r^2}{R^2}\Leftrightarrow R^2\geq 4r^2\Leftrightarrow R\geq 2r$$

Equality holds for $a=b=c$.