ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\geq\frac{6r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=\sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{\frac{2F}{2}}{\frac{F}{s-a}}=2\sum_{cyc}\frac{s-a}{a}=2\left(s\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{a}\right)=$$

$$=2\left(s\cdot\frac{ab+bc+ca}{abc}-3\right)=2\left(s\cdot\frac{s^2+r^2+4Rr}{4Rrs}-3\right)=$$

$$=2\left(\frac{s^2+r^2+4Rr-12Rr}{4Rr}\right)\overset{GERRETSEN}{\geq}$$

$$\geq 2\cdot\frac{16Rr-5r^2+r^2-8Rr}{4Rr}=\frac{8Rr-4r^2}{2Rr}=\frac{4R-2r}{R}\overset{EULER}{\geq}\frac{4\cdot 2r-2r}{R}=\frac{6r}{R}$$

Equality holds for $a=b=c$.