ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$GA^2+GB^2+GC^2 \le 3R^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$GA^2+GB^2+GC^2=\sum_{cyc} GA^2=\sum_{cyc}\left(\frac{2}{3}m_a\right)^2=\frac{4}{9}\sum_{cyc} m_a^2=$$

$$=\frac{4}{9}\cdot\frac{3}{4}(a^2+b^2+c^2)=\frac{1}{3}(a^2+b^2+c^2)=\frac{1}{3}\cdot 2(s^2-r^2-4Rr)\overset{GERRETSEN}{\le}$$

$$\le\frac{2}{3}(4R^2+3r^2+4Rr-r^2-4Rr)=\frac{2}{3}(4R^2+2r^2)=\frac{2}{3}\left(4R^2+2\cdot\frac{R^2}{4}\right)$$

$$=\frac{2}{3}\cdot\frac{18R^2}{4}=\frac{18R^2}{6}=3R^2$$

Equality holds for $a=b=c$.