ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a^2+b^2}{\sqrt{r_c}}\le 2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let }\sqrt{s-a}=x,\sqrt{s-b}=y,\sqrt{s-c}=z\ (\because (s-a),(s-b),(s-c)>0)$$

$$\text{and then}: s-a=x^2, s-b=y^2, s-c=z^2\Rightarrow 3s-2s=\sum_{cyc}x^2\text{ and so,}$$

$$a=y^2+z^2, b=z^2+x^2, c=x^2+y^2\ \therefore \sum_{cyc}\frac{b^2+c^2}{\sqrt{r_a}}\overset{?}{\le}2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}\Leftrightarrow$$

$$\sum_{cyc}\frac{x((z^2+x^2)^2+(x^2+y^2)^2)}{\sqrt{rs}}\overset{?}{\le}\sum_{cyc}\frac{(y^2+z^2)^2.\sqrt{2(y^2+z^2)}}{\sqrt{rs}}$$

$$\Leftrightarrow \sum_{cyc}\left((y^2+z^2)^2.\sqrt{2(y^2+z^2)}\right)\boxed{\underset{(*)}{\overset{?}{\ge}}}\sum_{cyc}\left(x((z^2+x^2)^2+(x^2+y^2)^2)\right)$$

$$\text{Now, }\sum_{cyc}\left((y^2+z^2)^2.\sqrt{2(y^2+z^2)}\right)\ge\sum_{cyc}\left((y^2+z^2)^2(y+z)\right)$$

$$=(y^2+z^2)^2(y+z)+(z^2+x^2)^2(z+x)+(x^2+y^2)^2(x+y)$$

$$=x((z^2+x^2)^2+(x^2+y^2)^2)+y((x^2+y^2)^2+(y^2+z^2)^2)+$$

$$z((y^2+z^2)^2+(z^2+x^2)^2)=\sum_{cyc}\left(x((z^2+x^2)^2+(x^2+y^2)^2)\right)\Rightarrow(*)\text{ is true}$$

$$\text{and so, }\sum_{cyc}\frac{a^2+b^2}{\sqrt{r_c}}\le 2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Jenish Rijal-Nepal

$$\textit{Here, we know}: \frac{1}{\sqrt{h_a}}=\sqrt{\frac{1}{2}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)}\overset{\overbrace{\text{Power Mean}}}{\ge}\frac{1}{2}\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\Leftrightarrow\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\le\frac{2}{\sqrt{h_a}}$$

$$\textit{Now, we have}: \sum_{cyc}\frac{a^2+b^2}{\sqrt{r_c}}=\sum_{cyc}a^2\left(\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\right)\le\sum_{cyc}a^2\left(\frac{2}{\sqrt{h_a}}\right)=2\sum_{cyc}\frac{a^2}{\sqrt{h_a}}$$

Equality holds if and only if the triangle is equilateral.