ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(r_a+r_b)^{c^2}.(r_b+r_c)^{a^2}.(r_a+r_c)^{b^2}\le(3R)^{9R^2}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$(r_a+r_b)^{c^2}.(r_b+r_c)^{a^2}.(r_a+r_c)^{b^2}\overbrace{\le}^{Weighted\ \ AM-GM}$$

$$\le\left(\frac{c^2(r_a+r_b)+a^2(r_b+r_c)+b^2(r_a+r_c)}{(a^2+b^2+c^2)}\right)^{\sum a^2}\le$$

$$In\ \Delta ABC\ wlog:\ \ a\le b\le c,\qquad r_a\le r_b\le r_c$$

$$\to\{r_a+r_b\le r_a+r_c\le r_b+r_c$$

$$\overbrace{\le}^{Chebyshev}\left(\frac{2(r_a+r_b+r_c).\frac{1}{3}(a^2+b^2+c^2)}{(a^2+b^2+c^2)}\right)^{\sum a^2}=$$

$$=\left(\frac{2}{3}(4R+r)\right)^{\sum a^2}\overbrace{\le}^{Euler}\left(\frac{2}{3}.\frac{9R}{2}\right)^{\sum a^2}\overbrace{\le}^{Leibniz}(3R)^{18R^2}$$

$$Equality\ \ holds\ \ if\ \ a=b=c$$