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In $a$ny Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{r_c^2}\ge 2\sum_{cyc}\frac{\sqrt{a}}{h_a^2}$$

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$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{r_c^2}\overset{?}{\ge}2\sum_{cyc}\frac{\sqrt{a}}{h_a^2}\Leftrightarrow\frac{(\sqrt{b}+\sqrt{c})(b+c-a)^2}{F^2}\overset{?}{\ge}2.\sum_{cyc}\frac{a^2.\sqrt{a}}{F^2}$$

$$\Leftrightarrow\sum_{cyc}\left((y+z)(y^2+z^2-x^2)^2\right)\overset{?}{\ge}2\sum_{cyc}x^5\ \left(\sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y+\sum_{cyc}xy^4\overset{?}{\ge}2xyz\left(\sum_{cyc}xy\right)\to\text{true}\because\sum_{cyc}x^4y+\sum_{cyc}xy^4=$$

$$\sum_{cyc}\left(z(x^4+y^4)\right)\overset{AM-GM}{\ge}2\sum_{cyc}x^2y^2z=2xyz\left(\sum_{cyc}xy\right)$$

$$\therefore\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{r_c^2}\ge 2\sum_{cyc}\frac{\sqrt{a}}{h_h^2}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equil}a\text{ter}al\ (QED)$$