ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(r_a)^{b^2+c^2}.(r_b)^{a^2+c^2}.(r_c)^{b^2+a^2} \le \left(\frac{3R}{2}\right)^{18R^2}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$(r_a)^{b^2+c^2}.(r_b)^{a^2+c^2}.(r_c)^{b^2+a^2} \overset{Weighted\ \ AM-GM}{\overbrace{\le}}$$

$$\le \left(\frac{r_a(b^2+c^2)+r_b(a^2+c^2)+r_c(b^2+a^2)}{2(a^2+b^2+c^2)}\right)^{2\sum a^2} \le$$

$$In\ \Delta ABC\ wlog:\ \ a \le b \le c\,, \qquad r_a \le r_b \le r_c$$

$$\rightarrow \{b^2+c^2 \ge a^2+c^2 \ge b^2+a^2$$

$$\overset{Chebyshev}{\overbrace{\le}} \left(\frac{\frac{1}{3}(r_a+r_b+r_c).2(a^2+b^2+c^2)}{2(a^2+b^2+c^2)}\right)^{2\sum a^2} =$$

$$= \left(\frac{1}{3}(4R+r)\right)^{2\sum a^2} \overset{Euler}{\overbrace{\le}} \left(\frac{1}{3}.\frac{9R}{2}\right)^{2\sum a^2} \overset{Leibniz}{\overbrace{\le}} \left(\frac{3R}{2}\right)^{18R^2}$$

$$Equality\ \ holds\ \ if\ a=b=c.$$