ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{a}}{r_a}\le\sum_{cyc}\frac{\sqrt{a}}{h_a}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Jenish Rijal-Nepal

$WLOG\ let's\ assume\ that\ a\ge b\ge c\ \Rightarrow \sqrt{a}\ge\sqrt{b}\ge\sqrt{c}\ \Rightarrow$

$$\frac{1}{r_a}\le\frac{1}{r_b}\le\frac{1}{r_c}\ and\ \frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}.$$

$$\sum_{cyc}\left(\sqrt{a}\cdot\frac{1}{r_a}\right)\overbrace{\le}^{Chebyshev}\frac{1}{3}\cdot\sum_{cyc}(\sqrt{a})\cdot\sum_{cyc}\left(\frac{1}{r_a}\right)\overbrace{\le}^{\sum\frac{1}{r_a}=\sum\frac{1}{h_a}}$$

$$\le\frac{1}{3}\cdot\sum_{cyc}(\sqrt{a})\cdot\sum_{cyc}\left(\frac{1}{h_a}\right)\overbrace{\le}^{Chebyshev}\sum_{cyc}\left(\sqrt{a}\cdot\frac{1}{h_a}\right)$$

$$\therefore\ \sum_{cyc}\frac{\sqrt{a}}{r_a}\le\sum_{cyc}\frac{\sqrt{a}}{h_a}$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$